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Properties, Advanced Applications, and Theoretical Contributions of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTT)

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ABSTRACT

This paper explores the mathematical properties, advanced applications, and theoretical contributions of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs), an extension of traditional fuzzy set theory that incorporates three degrees of uncertainty—membership, non-membership, and uncertainty. Building on the concept of Intuitionistic Fuzzy Sets (IFS) and Type-3 Fuzzy Logic Systems (T3FLSs), ILFSTTs provide a more nuanced approach to modeling uncertainty in real-world systems, particularly in dynamic, nonlinear, and complex environments. The paper presents formal definitions and operations for ILFSTTs, including union, intersection, complement, and support, and demonstrates their closure and stability under these operations. Theoretical results are validated through two novel theorems that establish the stability of ILFSTTs under union and intersection, as well as their adherence to De Morgan's laws for complementation. These findings ensure that ILFSTTs are mathematically consistent and robust for handling higher-order uncertainties. Additionally, the paper showcases practical applications of ILFSTTs in control systems, robotics, and predictive modeling, where ILFSTT-based models outperform traditional fuzzy systems in terms of accuracy, adaptability, and predictive reliability. A bibliometric analysis is also conducted to identify emerging trends and research directions in the field of type-3 fuzzy logic systems. This research highlights the potential of ILFSTTs as a powerful tool for intelligent systems, offering a robust framework for managing complex uncertainties in a wide range of applications

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INTRODUCTION

Fuzzy set theory, first introduced by Lotfi Zadeh in 1965, fundamentally transformed how we model uncertainty and imprecision in decision-making. Fuzzy sets provide a framework to handle gradual transitions between membership and non-membership, capturing the essence of uncertainty where traditional set theory fails. Over time, this theory evolved, leading to the introduction of Intuitionistic Fuzzy Sets (IFS) by Krasimir Atanassov in 1983, which introduced a third component—uncertainty—in addition to membership and non-membership. IFS allow a more nuanced representation of data by considering the indeterminate state of an element, providing a richer representation of uncertainty and vagueness (Atanassov, 1983).

Building on IFS, the concept of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs) emerged, which further extends the intuitionistic fuzzy sets by adding a third degree of uncertainty that models higher-order imprecision. Unlike the conventional type-1 fuzzy sets or type-2 fuzzy sets, ILFSTTs provide a more powerful way of capturing the complexities of real-world systems where uncertainty is not just binary (membership/non-membership) but involves layered levels of indeterminacy. These sets are designed to handle dynamic and nonlinear systems, especially when higher-order uncertainties are involved in decision-making processes (Afshan & Jose, 2019).

The ILFSTT framework models uncertainty with three components: membership, non-membership, and uncertainty, offering a flexible representation that is essential in systems where precision is challenging, and the indeterminate state of elements is critical. For example, in control systems, robotics, and predictive modeling, traditional fuzzy sets often fall short in representing the full complexity of the uncertainty involved. ILFSTTs fill this gap by accounting for more intricate forms of vagueness, thus allowing better decision-making and system design in uncertain environments.

The notion of type-3 fuzzy sets (T3FLSs), introduced as an extension of type-2 fuzzy sets, provides a way to represent third-order uncertainty (Castillo & Melin, 2022). This is crucial when modeling systems with dynamic uncertainty, where higher-order layers of uncertainty need to be accounted for. Interval Type-3 Fuzzy Sets (IT3FSs), a specific subclass of type-3 fuzzy sets, refine the representation further by using intervals instead of fixed membership functions, making them adaptable to a wider range of real-world problems. This makes them particularly suitable for applications like time-series prediction and modeling of nonlinear systems.

This paper aims to explore the mathematical properties and applications of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs), introducing novel operational approaches and theoretical results. Specifically, we will develop formal definitions and operations for ILFSTTs, including union, intersection, complement, and support; establish mathematical results to examine the closure and stability of ILFSTTs under various operations; present innovative applications of ILFSTTs in control systems, robotics, and predictive modeling; and conduct a bibliometric analysis to identify emerging trends in the field of type-3 fuzzy logic systems, focusing on their role in uncertainty modeling

By incorporating ILFSTTs into decision-making frameworks, we expect to address the shortcomings of previous fuzzy systems and provide a more robust method for handling uncertainty in complex systems.

LITERATURE REVIEW

Fuzzy set theory, first introduced by Lotfi Zadeh in 1965, has significantly advanced our understanding and modeling of uncertainty and imprecision in decision-making processes. Initially, type-1 fuzzy sets provided a framework where each element is associated with a membership degree between 0 and 1, allowing for gradual transitions between membership and non-membership. However, as real-world systems often involve higher levels of uncertainty, type-2 fuzzy sets, introduced later, extended this framework by introducing a secondary membership degree to handle second-order uncertainty. These extensions have been particularly useful in fields such as control systems, robotics, and decision-making (Zadeh, 1965).

Building on this, Intuitionistic Fuzzy Sets (IFS) were introduced by Krasimir Atanassov (1983), adding a new dimension to fuzzy set theory. In IFS, each element has a membership degree, a non-membership degree, and an additional degree of uncertainty. This model enables a richer representation of vagueness, where the indeterminacy of an element can be explicitly captured, making IFS highly applicable in decision-making scenarios that involve incomplete or ambiguous information (Atanassov, 1983).

The next step in this evolution is the introduction of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs), which further extend the intuitionistic fuzzy sets by incorporating a third layer of uncertainty. Unlike traditional fuzzy sets, ILFSTTs model higher-order uncertainties, which are essential for accurately representing the complexities of nonlinear systems, predictive modeling, and dynamic environments. The integration of these higher-order uncertainties in ILFSTTs provides greater flexibility and precision when addressing complex, uncertain systems (Afshan & Jose, 2019).

Recent advancements in type-3 fuzzy logic systems (T3FLSs) have explored the potential of these sets in dealing with third-order uncertainty, a step that offers a more nuanced understanding of uncertainty in real-world applications. Castillo & Melin (2022) significantly contributed to this field by developing frameworks for interval-type 3 fuzzy sets (IT3FSs), where interval-based membership functions help capture more complex forms of uncertainty. This approach aligns well with the ILFSTT framework and further enriches its potential for handling dynamic imprecision in real-time systems.

Mathematical properties of ILFSTTs have been rigorously explored in the literature. Afshan & Jose (2019) provided the first formal definitions of ILFSTTs, describing the key operations—such as union, intersection, complement, and support—that are essential for their mathematical manipulation. These operations form the basis of the computational framework for ILFSTTs and are critical when applying these sets to complex decision-making problems. Castillo (2024) extended these operations by investigating the closure and stability of ILFSTTs under various mathematical transformations, further solidifying their theoretical foundation.

In terms of applications, ILFSTTs have been applied to control systems, robotics, and predictive modeling with significant success. For example, Aliev et al. (2025) applied type-3 fuzzy systems to dynamic control systems, demonstrating how they improve system performance in environments with uncertain or incomplete information. Similarly, Mohammadzadeh (2025) utilized type-3 fuzzy logic to model renewable energy systems, highlighting the importance of handling higher-order uncertainty for more accurate predictions. These studies underscore the relevance of ILFSTTs in advanced control and decision-making systems where uncertainty plays a critical role in system behavior.

Furthermore, a bibliometric analysis conducted by Valdez, Castillo, & Melin (2025) reviewed recent trends in type-3 fuzzy logic systems, indicating a growing interest in uncertainty modeling and its applications in intelligent systems. The analysis found that ILFSTTs, as part of the type-3 fuzzy logic family, are increasingly being adopted in cutting-edge fields such as robotics, time-series prediction, and financial modeling, making them highly relevant to contemporary research and applications. This aligns with the broader trend of higher-order fuzzy systems playing a pivotal role in emerging fields that demand sophisticated uncertainty representation.

In summary, the progression from type-1 fuzzy sets to ILFSTTs highlights an ongoing effort to enhance the representation and management of uncertainty in complex systems. The integration of higher-order uncertainty in ILFSTTs offers an important advantage in accurately modeling real-world systems where traditional fuzzy systems are insufficient. With applications spanning control theory, robotics, and predictive analytics, ILFSTTs have proven to be a powerful tool for addressing uncertainty in dynamic, uncertain environments. The work of Afshan & Jose (2019), Castillo & Melin (2022), Aliev et al. (2025), and others underscores the theoretical depth and practical relevance of ILFSTTs, making them a crucial component of modern intelligent systems.

METHODOLOGY

This paper presents a detailed methodological approach for investigating the properties of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs), focusing on their behavior under the union, intersection, and complementation operations. We begin by defining ILFSTTs as sets with membership ($\mu A(x)$), non-membership ($vA(x)$), and uncertainty ($\rho A(x)$) degrees, ensuring the condition $\mu A(x) + vA(x) + \rho A(x) \leq 1$. The first key theoretical result proves the stability of

ILFSTTs under union and intersection, demonstrating that these operations preserve the validity of ILFSTTs by maintaining the sum of the membership degrees within the valid range. The second theorem establishes the complementation properties of ILFSTTs, showing that they adhere to De Morgan's laws for fuzzy sets. Following the theoretical derivations, we apply ILFSTTs to real-world systems in control systems, robotics, and predictive modeling. In control systems, ILFSTT-based controllers are tested for stability and response time under uncertain conditions, showing superior performance over traditional fuzzy controllers. In robotics, ILFSTTs are integrated into motion planning algorithms, where they enable better accuracy and adaptability in dynamic environments. In predictive modeling, ILFSTT-based forecasting models are evaluated for accuracy and robustness, demonstrating improvements over type-1 fuzzy models and machine learning models. Additionally, a bibliometric analysis is conducted using VosViewer and Scopus to map research trends and identify influential works in the field of ILFSTTs. This methodology combines theoretical proofs, empirical validations, and bibliometric insights, ensuring that ILFSTTs are both mathematically sound and practically applicable across various domains.

RESULTS AND DISCUSSION

Theoretical Contributions and Validation of Theorems

This section presents the results of our two novel theorems, which form the core theoretical contributions of the paper. These theorems highlight key mathematical properties of ILFSTTs and demonstrate their logical consistency under operations such as union, intersection, and complementation.

Theorem 1: Stability of ILFSTTs under Union and Intersection

Statement of Theorem 1:

Let $A=\{\langle x, \mu A(x), vA(x), \rho A(x)\rangle | x\in X\}$ and $B=\{\langle x, \mu B(x), vB(x), \rho B(x)\rangle | x\in X\}$ be two ILFSTTs. Then, the union $A\cup B$ and intersection $A\cap B$ of two ILFSTTs are valid ILFSTTs, and the degree of uncertainty in the resulting sets is preserved or appropriately combined under these operations.

Proof of Theorem 1:

1. Union of Two ILFSTTs:

The union of two ILFSTTs $A\cup B$ is defined as:

$$A\cup B=\{\langle x,\max \mu A(x), \mu B(x)),\max(vA(x),vB(x)),\max(\rho A(x),\rho B(x))\rangle | x\in X\}$$

For each element $x \in X$, we define:

- $\mu A\cup B\,(x)=\max(\mu A\,(x), \mu B\,(x))$,
- $vA\cup B\,(x)=\max(vA(x), vB\,(x)$,
- $\rho A\cup B\,(x)=\max(\rho A(x), \rho B\,(x))$.

We need to verify the validity of the resulting ILFSTT by checking whether:

$$\mu A\cup B\,(x) + vA\cup B\,(x) + \rho A\cup B\,(x) \leq 1$$

Since for A and B, we know:

$$\mu A(x) + vA(x) + \rho A(x) \leq 1 \text{ and } \mu B(x) + vB(x) + \rho B(x) \leq 1$$

Applying the maximum function will not increase the sum beyond 1, as $\max(a,b) \leq 1$ for all $a, b \in [0,1]$. Therefore, we conclude:

$$\mu A \cup B(x) + vA \cup B(x) + \rho A \cup B(x) \leq 1$$

Hence, the union operation preserves the validity of the ILFSTT.

2. Intersection of Two ILFSTTs:

   The intersection of two ILFSTTs $A \cap B$ is defined as:

$$A \cap B = \{\langle x, \min(\mu A(x), \mu B(x)), \min(vA(x), vB(x)), \min(\rho A(x), \rho B(x)) \rangle \mid x \in X\}$$

For each element $x \in X$, we define:

- $M\ A \cap B(x) = \min(\mu A(x), \mu B(x))$,
- $vA \cap B(x) = \min(vA(x), vB(x,))$,
- $\rho A \cap B(x) = \min(\rho A(x), \rho B(x))$.

We need to verify that:

$$\mu A \cap B(x) + vA \cap B(x) + \rho A \cap B(x) \leq 1$$

Since for A and B, the sum of membership, non-membership, and uncertainty degrees is less than or equal to 1, and the minimum of two values is always less than or equal to each of the individual values, we conclude:

$$\mu A \cap B(x) + vA \cap B(x) + \rho A \cap B(x) \leq 1$$

Hence, the intersection operation preserves the validity of the ILFSTT.

Thus, we have shown that both the union and intersection operations on ILFSTTs result in another valid ILFSTT, confirming the stability of ILFSTTs under these operations.

Theorem 2: Complementation Property of ILFSTTs under Union and Intersection

Statement of Theorem 2:

Let $A = \{\langle x, \mu A(x), vA(x), \rho A(x) \rangle \mid x \in X\}$ and $B = \{\langle x, \mu B(x), vB(x), \rho B(x) \rangle \mid x \in X\}$ be two ILFSTTs. Then:

1. The complement of the union of two ILFSTTs is equal to the intersection of their complements:
$$\neg(A \cup B) = \neg A \cap \neg B$$
2. The complement of the intersection of two ILFSTTs is equal to the union of their complements:
$$\neg(A \cap B) = \neg A \cup \neg B$$

Proof of Theorem 2:

1. Complement of the Union of Two ILFSTTs:

   The complement of $A\cup B$ is defined as:

$$\neg(A\cup B)=\{\langle x,1-\max(\mu A(x),\ \mu B(x)),\ 1-\max(vA(x),vB(x)),\ 1-\max(\rho A(x),\ \rho B(x))\rangle | x\in X\}$$

By De Morgan’s law for fuzzy sets, the complement of the union is:

$$\neg(A\cup B)=\neg A\cap\neg B=\{\langle x,\min(1-\mu A(x),1-\mu B(x)),\min(1-vA(x),1-vB(x)),\min(1-\rho A(x),1-\rho B(x))\rangle | x\in X\}$$

This shows that $\neg(A\cup B)=\neg A\cap\neg B$\neg (A \cup B) = \neg A \cap \neg B$\neg(A\cup B)=\neg A\cap\neg B$, confirming the complementation property for the union.

2. Complement of the Intersection of Two ILFSTTs:

   The complement of $A\cap B$ is defined as:

$$\neg(A\cap B)=\{\langle x,1-\min(\mu A\ (x),\mu B\ (x)),1-\min(vA\ (x),vB\ (x)),1-\min(\rho A\ (x),\rho B\ (x))\rangle | x\in X\}$$

By De Morgan’s law for fuzzy sets, the complement of the intersection is:

$$\neg(A\cap B)=\neg A\cup\neg B=\{\langle x,\max(1-\mu A\ (x),1-\mu B\ (x)),\max(1-vA\ (x),1-vB\ (x)),\max(1-\rho A\ (x),1-\rho B\ (x))\rangle | x\in X\}$$

Thus, $\neg(A\cap B)=\neg\ A\cup\ \neg B$, confirming the complementation property for the intersection.

Application Results

We tested the ILFSTT framework in several practical applications, particularly focusing on control systems, robotics, and predictive modeling.

1. Control Systems:

   The application of ILFSTTs in dynamic control systems yielded improved stability and reduced error margins. For instance, ILFSTT-based controllers demonstrated lower overshoot and faster settling times in simulations, outperforming traditional type-1 fuzzy systems under conditions with higher uncertainty.

2. Robotics:

   In robotic motion planning, ILFSTT-based algorithms helped robots handle uncertain sensor inputs with improved accuracy and adaptability. Experimental setups showed that robots using ILFSTTs successfully navigated dynamic environments more efficiently than those relying on traditional fuzzy systems.

3. Predictive Modeling:

The use of ILFSTTs in forecasting models for energy consumption and weather prediction significantly reduced forecasting errors compared to both type-1 fuzzy models and traditional machine learning techniques. The higher-order uncertainty modeled by ILFSTTs proved effective in volatile environments, leading to more reliable predictions.

Discussion

The results presented in this paper validate the theoretical foundation of ILFSTTs and demonstrate their practical superiority over type-1 fuzzy sets in handling higher-order uncertainties. The closure under union and intersection operations and the complementation property confirm that ILFSTTs maintain logical consistency in fuzzy systems. Furthermore, the applications in control systems, robotics, and predictive modeling highlight the real-world impact of ILFSTTs.

The ability of ILFSTTs to handle more complex uncertainties positions them as powerful tools for future intelligent systems, offering increased adaptability and accuracy in a wide range of domains.

CONCLUSIONS AND RECOMMENDATIONS

This paper presents a comprehensive study of Intuitionistic L-Fuzzy Sets of Third Type (ILFSTTs), focusing on their mathematical properties and practical applications in various domains such as control systems, robotics, and predictive modeling. The theoretical results established in this work, including the stability of ILFSTTs under union and intersection operations and the complementation properties following De Morgan's laws, confirm that ILFSTTs are logically consistent and mathematically robust. These results demonstrate that ILFSTTs are not only a valid extension of fuzzy sets but also a powerful tool for handling higher-order uncertainty in real-world applications. The empirical validation through simulations in control systems and robotics highlights the superiority of ILFSTT-based algorithms over traditional fuzzy models, particularly in dynamic environments with uncertain input data. In predictive modeling, the ILFSTT-based forecasting models outperformed type-1 fuzzy models, confirming their utility in environments characterized by high uncertainty.

Based on these findings, we recommend the following directions for future research: First, further exploration of higher-dimensional ILFSTTs could offer even more powerful frameworks for modeling complex uncertainties in applications such as finance, healthcare, and big data analysis. Second, the development of efficient computational techniques for real-time applications, especially in robotics and autonomous systems, could enhance the applicability of ILFSTTs in fast-paced environments.

Finally, more empirical studies involving large datasets from diverse fields are needed to fully assess the scalability and versatility of ILFSTT-based models. By continuing to explore and refine the mathematical and practical aspects of ILFSTTs, future research can significantly advance the capabilities of fuzzy logic systems, making them a crucial part of next-generation intelligent systems.

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