



Nearly Essential Submodule

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ABSTRACT

This paper investigates the concept of a nearly essential submodule of an R -module Y , together with its fundamental properties and its relationship with essential submodules. The aim of this work is to establish several properties of nearly essential submodules and to clarify their relationship with essential submodules. In particular, it is shown that each essential submodule is nearly essential, whereas the opposite does not hold overall. Several results concerning the behavior of nearly essential submodules under inclusion, intersection, and sum are established. In particular, if a submodule contains a nearly essential submodule, then it is itself nearly essential. Conditions under which the intersection of two submodules is nearly essential are also investigated. Moreover, some results concerning essential and nearly essential submodules are obtained in relation to the Jacobson radical $J(Y)$. Furthermore, the relationship between nearly essential submodules and essential submodules in injective modules is studied. Thus, the obtained results demonstrate that the notion of a nearly essential submodule provides a generalization of the concept of an essential submodule.

INTRODUCTION

This is all rings research are commutative ring with a personality and all modules are unified left R -modules. Allow Y be an R -module. A submodule ω of Y is named an essential submodule of Y if $\omega \cap D \neq 0$ for every submodule D of Y [1]. In this work we introduce the concept of nearly essential submodule, which is a generalization of essential submodule. A submodule ω of Y is named a nearly essential submodule of Y if $\omega \cap J(Y) \neq 0$. Every essential submodule in an R -module Y is nearly essential submodule. However, the converse does not hold in general. This distinction makes nearly essential submodule a natural object for further investigation and raises questions concerning their behavior under the usual operations on submodules. In this paper, we study several fundamental properties of nearly essential submodules. First, we establish basic results concerning inclusion and show that if ω_1 is a nearly essential submodule of Y and $\omega_1 \subseteq \omega_2$, next ω_2 is also a nearly essential submodule of Y . We also investigate conditions under which the intersection of two submodules is nearly essential. In particular, suitable conditions involving the Jacobson radical are established to guarantee that $\omega_1 \cap \omega_2$ is nearly essential. Let ω be a submodule in an R -module Y , and S be a multiplicative set of R . $\omega(S) = \{x \in Y : \exists t \in S, tx \in \omega\}$, $\omega(S)$ is a R -submodule of Y contains ω [2]. We define a closure of a submodule ω , denoted by $cl(\omega) = \{y \in Y : [\omega : (y)] \text{ essential in } R\}$. $cl(\omega)$ is a submodule of Y , and $\omega \subseteq cl(\omega)$ [3]. Furthermore, we examine the relationship between nearly essential submodules and essential submodules. Several examples are presented to illustrate the difference between these two concepts and to show that the converse of some natural implications does not hold in general. We also investigate the behavior of nearly essential submodules with respect to sums and other constructions. A further objective of this paper is to study nearly essential submodules in an environment of injective module. Recall that an R -module Y is an injective if for every R -monomorphism $f : \mu \rightarrow \mu'$ and every R -monomorphism $g : Y \rightarrow \mu'$, there exists an R -monomorphism $h : Y \rightarrow \mu$ such that the following diagram is commute that is $f \circ h = g$ [4]. This observation enables us to establish an important characterization for an injective R -module Y , a submodule ω is essential submodule in Y if and only if ω is nearly essential submodule in Y . Hence, although nearly essential submodules constitute a weaker notion in general, the two notions coincide for injective modules. The main purpose of this paper is therefore to develop the basic theory of nearly essential submodules, establish their fundamental properties, investigate their relationships with essential submodules, and obtain results that clarify the behavior of this notion in important classes of R -modules, particularly injective modules.

LITERATURE REVIEW

Submodules are important structures in module theory because they provide a framework for analyzing the internal properties and relationships of modules. In particular, essential submodules are submodules that have a nonzero intersection with every nonzero submodule. This concept is useful in studying module extensions, embeddings, and structural properties.

The concept of nearly essential submodules extends the idea of essential submodules by considering conditions under which a submodule maintains a sufficiently strong relationship with other nonzero submodules. Previous studies on essential submodules and related module-theoretic concepts provide the theoretical foundation for examining the properties of nearly essential submodules.

Based on these concepts, this study focuses on the structural properties of nearly essential submodules and their relationship with essentiality in module theory. The discussion is expected to contribute to a clearer characterization of nearly essential submodules and provide a basis for further studies in the algebraic structure of modules.

METHODOLOGY

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RESEARCH RESULT AND DISCUSSION

Basic Properties of Nearly Essential Submodule

Definition

A submodule ω of a R -module Y is named Nearly essential submodule if $\omega \cap J(Y) \neq 0$ where $J(Y)$ is the intersection of all maximal submodule of Y .

An ideal I is called Nearly essential ideal $I \cap J(R) \neq 0$.

Remarks and Examples

1. It is evident that each essential submodule of an R -module Y is named Nearly essential submodule. However, the opposite is not true. As the example that follows explains this:
The submodule $(\bar{3})$ in the Z -module Z_{12} is Nearly essential submodule, given that $(\bar{3}) \cap J(Z_{12}) = (\bar{6})$. But $(\bar{3})$ isn't essential submodule of Z -module Z_{12} , because $(\bar{3}) \cap (\bar{4}) = 0$.
2. A submodule of Nearly essential submodule is not necessary Nearly essential submodule. As the example that follows explains this:
The submodule $(\bar{3})$ in the Z -module Z_{18} is Nearly essential submodule, but the submodule $(\bar{9})$ where $(\bar{9}) \subseteq (\bar{3})$ is not Nearly essential submodule of Z_{18} , since $(\bar{3}) \cap J(Z_{12}) = 0$.
3. Each submodule of $J(Y)$ is Nearly essential submodule.
4. Each submodule of the Z -module Z is not Nearly essential submodule .
5. In semi-simple module each submodule is not Nearly essential submodule .
6. Every proper submodule of the Z -module Z_{p^∞} .

Theorem : Let us Y be a R -module, and ω_1, ω_2 are submodules of Y such that $\omega_1 \subseteq \omega_2$. If ω_1 is Nearly essential submodule of Y , then ω_2 is Nearly essential submodule of Y .

Proof : Given that ω_1 is Nearly essential submodule of Y , next $\omega_1 \cap J(Y) \neq 0$. But $\omega_1 \subseteq \omega_2$ it follows that $\omega_2 \cap J(Y) \neq 0$. Hence ω_2 is Nearly essential submodule of Y . As a direct consequence of the above theorem we get the following corollary:

Corollary: Let Y be a R -module, and ω_1, ω_2 are submodules of Y using $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y , next ω_1 and ω_2 are Nearly essential submodules of Y

The opposite of corollary 2.4 is mostly untrue, for example, at the Z -module Z_{36} the submodules $(\bar{12})$ and $(\bar{18})$ are Nearly essential submodules of Z_{36} , because $(\bar{12}) \cap J(Z_{36}) \neq 0$ and $(\bar{18}) \cap J(Z_{36}) \neq 0$, where $J(Z_{36}) = (\bar{6})$. But $(\bar{12}) \cap (\bar{18}) = 0$ is not Nearly essential submodule of Z_{36} , because $(\bar{0}) \cap J(Z_{36}) = 0$.

Now we gives a partial converse of corollary 2.4.

Proposition: Let us Y be a R -module, and ω_1, ω_2 are submodules of Y such that $J(Y) \subseteq \omega_2$ and $J(\omega_2) \subseteq J(Y)$ If ω_1 is Nearly essential submodule of ω_2 then $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .

Proof: Given that ω_1 is Nearly essential submodule of ω_2 , by Definition 2.1 we have $\omega_1 \cap J(\omega_2) \neq 0$. By the additional assumption $J(\omega_2) \subseteq J(Y)$, it follows that $\omega_1 \cap J(\omega_2) \subseteq \omega_1 \cap J(Y)$. Therefore, $\omega_1 \cap J(Y) \neq 0$. On the other hand, since $J(Y) \subseteq \omega_2$, we obtain $(\omega_1 \cap \omega_2) \cap J(Y) = \omega_1 \cap (\omega_2 \cap J(Y)) = \omega_1 \cap J(Y)$. Hence $(\omega_1 \cap \omega_2) \cap J(Y) \neq 0$. Therefore $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .

Proposition: Let us Y be a R -module, and ω_1, ω_2 are submodules of Y so as to ω_1 is essential submodule of Y and ω_2 is Nearly essential submodule of Y . Next $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .

Proof: Given that ω_2 is Nearly essential submodule of Y , then $(\omega_2 \cap J(Y)) \neq 0$. Since ω_1 is essential submodule of Y , then $\omega_1 \cap (\omega_2 \cap J(Y)) \neq 0$, and so we get $(\omega_1 \cap \omega_2) \cap J(Y) \neq 0$, which implies that $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .

From thermo 2.3 we get the following corollaries:

Corollary : Let Y be an R -module, and ω is Nearly essential submodule of Y . Then:

1. $\omega(S)$ is Nearly essential submodule of Y .
2. $cl(\omega)$ is Nearly essential submodule of Y .
3. $\sqrt{\omega}$ is Nearly essential submodule of Y .

Corollary 2.8: Let us Y be an R -module, and ω_1, ω_2 are submodules of Y such that either ω_1 or ω_2 is Nearly essential submodule of Y , next $\omega_1 + \omega_2$ is Nearly essential submodule of Y .

The following lemma is necessary in order to demonstrate the following claim.

Lemma : If a R -module Y is an injective, next $J(Y) = Y$.

Proposition: Let us Y be an injective R -module and ω is a submodule of Y , next ω is an essential submodule of Y iff ω is Nearly essential submodule of Y .

Proof:

($\Rightarrow$) Direct by Remark and Examples 2.2 (1).

($\Leftarrow$) Suppose that ω is Nearly essential submodule of Y , next $\omega \cap J(Y) \neq 0$. But Y injective R -module, it follows that by lemma 2.9 $J(Y) = Y$, so we get $\omega \cap Y \neq 0$. Hence ω is an essential submodule of Y .

CONCLUSIONS AND RECOMMENDATIONS

Conclusion

- We will present the most important propositions in this research:
- Let Y be a R -module, and ω_1, ω_2 are submodules of Y using $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y , then ω_1 and ω_2 are Nearly essential submodules of Y .
- Let Y be a R -module, and ω_1, ω_2 are submodules of Y such that $J(Y) \subseteq \omega_2$ and $J(\omega_2) \subseteq J(Y)$ If ω_1 is Nearly essential submodule of ω_2 next $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .
- Let us Y be a R -module, and ω_1, ω_2 are submodules of Y so as to ω_1 is essential submodule of Y and ω_2 is Nearly essential submodule of Y . Next $\omega_1 \cap \omega_2$ is Nearly essential submodule of Y .
- Let Y be an injective R -module and ω is a submodule of Y , then ω it is essential submodule of Y iff ω is Nearly essential submodule of Y .

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